ULAM–HYERS–MITTAG-LEFFLER STABILITY FOR FRACTIONAL-ORDER DELAY DIFFERENTIAL EQUATIONS

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In this paper, we present results on the existence, uniqueness, and Ulam–Hyers–Mittag-Leffler stability of solutions to a class of fractional-order delay differential equations. We use the Picard operator method and a generalized Gronwall inequality involved in Riemann–Liouville fractiona lintegral .Finally, we give two examples to illustrate our main theorems.

Keywords

Fractional-order delay differential equations; Solutions; Existence; Stability.

1. Introduction

Fractional-order differential equations are important since the ir non local property is suit-able to characterize memory phenomena in economic, control, and materials sciences. Existence, stability, and control theory to fractional differential equations was investigated in[1–21].In particular, the Ulam-type stability of delay differential equations was investigated in[22–30]. In[22], results for a delay differential equation were obtained using the Picard operator method, and in[23] the authors adopted a similar approach to establish the existence and uniqueness results for a Caputo-type fractional-order delay differential equation. In[31,32],the authors gave stability and numerical schemes for two classes of fractional equations.So usaand Oliveira[33] proposed the ψ-Hilfer fractional differentiation operator and established ψ-Hilfer fractional differential equations. In[24] the authors studied the Ulam–Hyers stability and the Ulam–Hyers–Rassias stability of ψ-Hilfer fractional integro-differential equations via the Banach fixed point method, and in [28] the author discussed the existence and uniqueness of solutions and Ulam–Hyers and Ulam– Hyers–Rassias stabilities for ψ-Hilfer nonlinear fractional differential equations via a generalized Gronwall inequality (see[34]).

Motivated by [23, 24, 28], we consider the ψ -Hilfer fractional differential equation

$$\begin{cases} {}^{H}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\, x(\tau) = f\left(\tau, x(\tau), x\left(g(\tau)\right)\right), & \tau \in I = (0,d], \\ I_{0^+}^{1-\gamma;\psi}\, x(0^+) = x_0 \in \mathbb{R}, \\ x(\tau) = \varphi(\tau), & \tau \in [-h, 0], \end{cases} \quad (1)$$

where ${}^{H}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}(.)$ is the ψ -Hilfer fractional derivative (see Definition 2.1) of order $0 < \alpha \leq 1$ and type

$0 < \beta \leq 1$, $I_{0^+}^{1-\gamma;\psi}(.)$ is the Riemann-Liouville fractional integral of order $1-\gamma$, $\gamma = \alpha + \beta(1-\alpha)$ with respect to the function ψ(see [2], and $f: I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function.

We establish the existence and uniqueness of solutions for (1) using the Picard operator approach in a weight function space. We also introduce and present Ulam-Hyers-Mittag-Leffler stability of soltions to (1).

2. Preliminaries

We collect the basic definitions of the ψ -Riemann–Liouville fractional integral, the ψ - Hilfer fractional derivative, and the standard Picard operator and an abstract Gronwall lemma.

Let [c, d] (o <c <d < ∞) bea finite interval on $\mathbb{R}^+$, and let C[c, d] be the space of continuous function g : [c, d] → ℝ with norm

$$\|g\|_{C[c,d]} = \max_{c\le x\le d}|g(x)|.$$

The weighted space $C_{1-\gamma;\psi}[c,d]$ of continuous g on (c, d] is defined by (see [24])

$$C_{1-\gamma;\psi}[c,d] = \left\{g: (c,d] \to \mathbb{R};\ \left(\psi(x)-\psi(c)\right)^{1-\gamma} g(x) \in C[c,d]\right\}, \qquad 0 \leq \gamma < 1,$$

with norm

$$\|g\|_{C_{1-\gamma;\psi}}[c,d] = \max_{x\in[c,d]}|(\psi(x) - \psi(c))^{1-\gamma}\, g(x)|.$$

or

$$\|g\|_B \coloneqq \max_{\tau\in[c,d]}|(\psi(\tau) - \psi(c))^{1-\gamma}\, g(\tau)|e^{-\theta(\psi(\tau)-\psi(c))}, \quad \theta > 0.$$

Definition 2.1

Let (c,d)($-\infty \le c<d \le \infty$) be a finite or infinite interval of the real line R,and let $\alpha>0$. In addition, let $\psi(x)$ be an increasing and positive emonotone function on(c,d] having a continuous derivative $\psi'(x)$ on (c,d). The lef t-and right-sided fractional integrals of a function g with respect to a function ψ on [c,d] aredefinedby

$$I_{c^+}^{\alpha:\psi} g(x) = \frac{1}{\Gamma(\alpha)} \int_c^x \psi'(\tau)\big(\psi(x) - \psi(\tau)\big)^{\alpha-1} g(\tau)\, dt,$$

$$I_{d^-}^{\alpha:\psi} g(x) = \frac{1}{\Gamma(\alpha)} \int_x^d \psi'(\tau)\big(\psi(\tau) - \psi(x)\big)^{\alpha-1} g(\tau)\, dt,$$

Respectively; here Γ is the gamma function.

Definition 2.2

Let $n-1<\alpha<n$ with $n \in \mathbb{N}$, and let $f, \psi \in C^n[c,d]$ be two functions such that ψ is increasing and $\psi'(x) \neq 0$ for all $x \in [c,d]$. The left-side ψ-Hilfer fractional derivative ${}^{H}\mathbb{D}_{c^+}^{\alpha,\beta;\psi}(.)$of a function g of order α and type $0 \le \beta \le 1$ is defined by

$${}^{H}\mathbb{D}_{c^+}^{\alpha,\beta;\psi} g(x) = I_{c^+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{c^+}^{(1-\beta)(n-\alpha);\psi}\, g(x).$$

The right-sided ψ $-$Hilfer fractional derivative is defined in an analogous way.

Theorem 2.3

If $g \in C^1[c,d], 0<c<1$, and $0 \le \beta \le 1$, then

$$I_{c^+}^{\alpha;\psi}\, {}^{H}\mathbb{D}_{c^+}^{\alpha,\beta;\psi} g(x) = g(x) - \frac{((\psi(x)-\psi(c))^{\gamma-1}}{\Gamma(\gamma)} I_{c^+}^{(1-\beta)(n-\alpha);\psi} g(c).$$

Let $I=[c,d]$. For $f \in C(I \times \mathbb{R}^2, \mathbb{R})$ and $\epsilon>0$, we consider the equations

$${}^{H}\mathbb{D}_{0^+}^{\alpha,\beta;\psi} x(\tau) = f\Big(\tau, x(\tau), x\big(g(\tau)\big)\Big), \quad \tau \in (0,d], \tag{2}$$

$$I_{0^+}^{1-\gamma;\psi}\, x(0^+) = x_0, \tag{3}$$

$$x(\tau) = \varphi(\tau), \;\; \tau \in [-h, 0], \tag{4}$$

and the inequality

$$\left| {}^{H}\mathbb{D}_{c^+}^{\alpha,\beta;\psi} x(\tau) - f\Big(\tau, x(\tau), x\big(g(\tau)\big)\Big) \right| \le \varepsilon E_\alpha\big(\psi(\tau) - \psi(0)\big)^\alpha, \tau \in (0,d], \tag{5}$$

where E_α is the Mittag – Leffler function [2] defined by

$$E_\alpha(x) := \sum_{i=0}^{\infty} \frac{x^i}{\Gamma(i\alpha + 1)}, \quad x \in \mathbb{C}, \Re(\alpha) > 0. \tag{6}$$

Motivated by [23, Lemma 2.4], we introduce the following concept.

Definition 2.5

Equation (2) is Ulam–Hyers–Mittag-Leffler stable with respect to $E_\alpha((\psi(\tau) - \psi(0))^\alpha)$ if there exists $c\mathbb{E}_\alpha > 0$ such that, for each $\varepsilon > 0$ and each solution $y \in C([-h,d],\mathbb{R})$ to (5), there exists a solution $x \in C([-h,d],\mathbb{R})$ to (2) with

$$|y(\tau) - x(\tau)| \le c\mathbb{E}_\alpha \varepsilon E_\alpha\big(\psi(\tau) - \psi(0)\big)^\alpha, \ \ \tau \in [-h, d],$$

Remark 2.6

A function $x \in C([-h,d],\mathbb{R})$ is a solution of inequality (5) if and only if there exists a function $\tilde{h} \in C([-h,d],\mathbb{R})$ (which depends on x) such that

(i) $\left|\tilde{h}(\tau)\right| \le \varepsilon E_\alpha((\psi(\tau) - \psi(0))^\alpha, \tau \in [-h, d],$

(ii) ${}^{H}\mathbb{D}_{o^+}^{\alpha,\beta;\psi} x(\tau) = f\left(\tau, x(\tau), x\big(g(\tau)\big)\right) + \tilde{h}(\tau), \ \tau \in (0, d]$

Definition 2.7

Let (Y, ρ) be a metric space. Now $T: Y \to Y$ is a Picard operator if there exists

$y^* \in Y$ such that

(i) $F_T = y^*$ where $F_T = \{y \in Y: T(y) = y\}$ is the fixed point set of T;

(ii) the sequence $(T^n(y_0))_{n\in\mathbb{N}}$ converges to y^* for all $y_0 \in Y$.

Lemma 2.8

Let $(Y, \rho, \le)$ be an ordered metric space, and let T: $Y \to Y$ be an increasing Picard operator with $F_T = \{y_T^*\}$. Then $y \in Y, \ y \le T(y)$ implies $y \le y_T^*$.

From Theorems 2.3 and 2.4 we have the following:

Lemma 2.9

Let $f: I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then (2) is equivalent to

$x(\tau) = \frac{((\psi(x)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\alpha-1} f\Big(s, x(s), x\big(g(s)\big)\Big)\, ds.$ (7)

Remark 2.10

Let $y \in C(I, \mathbb{R})$ bea solution of inequality (5). Then y is a solution of the following integral inequality:

$$\left| y(\tau) - \frac{((\psi(\tau) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 - \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\alpha-1} f\Big(s, y(s), y\big(g(s)\big)\Big)\, ds \right|$$

$$\leq \frac{\varepsilon}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\alpha-1} \mathbb{E}_\alpha\big(\psi(s) - \psi(0)\big)^\alpha)\, ds$$

$$= \frac{\varepsilon}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\alpha-1} \sum_{k=0}^\infty \frac{\big(\psi(s) - \psi(0)\big)^{k\alpha}}{\Gamma(k\alpha + 1)}\, ds$$

$$= \frac{\varepsilon}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \int_0^\tau \big(\psi(\tau) - \psi(s)\big)^{\alpha-1} \big(\psi(s) - \psi(0)\big)^{k\alpha}\, d\psi(s)$$

$$= \frac{\varepsilon}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \int_0^{\psi(\tau)-\psi(0)} (\psi(\tau) - \psi(0) - u)^{\alpha-1} u^{k\alpha}\, du$$

(let u = $\psi(s) - \psi(0)$)

$$= \frac{\varepsilon}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \big(\psi(\tau) - \psi(0)\big)^{\alpha-1} \int_0^{\psi(\tau)-\psi(0)} \left(1 - \frac{u}{\psi(\tau) - \psi(0)}\right)^{\alpha-1} u^{k\alpha}\, du$$

$$= \frac{\varepsilon}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \big(\psi(\tau) - \psi(0)\big)^{(k+1)\alpha} \int_0^1 v^{k\alpha}\, dv$$

$$\left(let\ v = \frac{u}{\psi(\tau) - \psi(0)}\right)$$

$$= \frac{\varepsilon}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \big(\psi(\tau) - \psi(0)\big)^{(k+1)\alpha} \frac{\Gamma(k\alpha + 1)\Gamma(\alpha)}{\Gamma((k + 1)\alpha + 1)}$$

$$\leq \varepsilon \sum_{n=0}^{\infty} \frac{\left(\psi(\tau)-\psi(0)\right)^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$= \varepsilon \mathbb{E}_{\alpha}\left(\left(\psi(\tau)-\psi(0)\right)^{\alpha}\right) \ .$$

Lemma 2.11

Let $\alpha > 0$, and let $\psi \in C^1((0,d],\mathbb{R})$ be a function such that ψ is increasing and $\psi'(\tau) \neq 0$ for all $\tau \in (0,d]$. Suppose that $d \geq 0, z$ is a nonnegative function locally integrable on (0, d], and w is nonegative integrable on (0, d] with

$$w(\tau) \leq z(\tau) + k\int_0^{\tau} \psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1} w(s)\, ds, \qquad \tau \in (0,d].$$

Then

$$w(\tau) \leq z(\tau) + \int_0^{\tau} \sum_{n=0}^{\infty} \frac{[k\Gamma(\alpha)]^n}{\Gamma(n\alpha)} \psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1} z(s)\, ds,$$
$$\tau \in (0,d].$$

Remark 2.12

Under the hypotheses of Lemma 2.11, let z be a non decreasing function on (0, d]. Then we have

$$w(\tau) \leq z(\tau)\mathbb{E}_{\alpha}(k\Gamma(\alpha)[\psi(\tau)-\psi(0)]^{\alpha}), \tau \in (0,d],$$

Where $\mathbb{E}_{\alpha}$ is the Mittag - Leffler function defined by (6) .

3. Main results

In this section, we establish the existence, uniqueness, and Ulam-Hyers-Leffler stability.

We impose the following conditions.

(H_1) $f \in C(I \times \mathbb{R}^2, \mathbb{R}), g \in C(I, [-h,d]), g(\tau) \leq \tau, h > 0$.

(H_2) There exists $L_f > 0$ such that

$$|f(\tau,u_1,u_2) - f(\tau,v_1,v_2)| \leq L_f \sum_{i=1}^{2} |u_i - v_i| \qquad for\ all\ \tau \in I, u_i, v_i \in \mathbb{R}, \quad i = 1,2 \ .$$

(H_3) We have the inequality

$$\frac{2L_f\Gamma(\gamma)(\psi(d)-\psi(0))^{\alpha}}{\Gamma(\gamma+\alpha)} < 1 \ .$$

Theorem 3.1

Assume that (H_1), (H_2), and (H_3) are satisfied. Then

(i) (2) – (4) has a unique solution in $C[-h,d] \cap C_{1-\gamma;\psi}[c,d]$.

(ii) (2) is Ulam – Hyers – Mittag – Leffler stable.

Proof

From Lemma 2.9 we get that (2) – (4) is equivalent to the following system:

$$y(\tau) = \begin{cases} \varphi(\tau), & \tau \in [-h,0], \\ \frac{(\psi(\tau)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 \\ + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} f\left(s, y(s), y(g(s))\right) ds, & \tau \in (0,d]. \end{cases} \tag{8}$$

The existence of a solution for (8) can be turned into a fixed point problem in $X \coloneqq C[-h,d]$ for the operator T_f defined by

$$T_f(x)(\tau) = \begin{cases} \varphi(\tau), & \tau \in [-h,0], \\ \frac{(\psi(\tau)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 \\ + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} f\left(s, y(s), y(g(s))\right) ds, & \tau \in (0,d]. \end{cases} \tag{9}$$

Note that for any continuous function f, T_f is also continuous. Indeed,

$$\left|T_f(x)(\tau) - T_f(x)(\tau_0)\right|$$

$$= \Bigg| \frac{(\psi(\tau) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 + + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1} f\left(s, y(s), y(g(s))\right) ds - \frac{(\psi(\tau_0) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 - \frac{1}{\Gamma(\alpha)} \int_0^{\tau_0} \psi'(s)(\psi(\tau_0) - \psi(s))^{\alpha-1} f\left(s, y(s), y(g(s))\right) ds \Bigg| \to 0$$

as $\tau \to \tau_0$.

Next, we show T_f defined in (9) is a contraction mapping on $X \coloneqq C[-h, d]$ with respect to $\| . \|_{C_{1-\gamma;\psi}}[0, d]$. Consider $T_f: X \to X$ defined in (9). For $\tau \in [-h, 0]$, we have

$$\left|T_f(x)(\tau) - T_f(y)(\tau)\right| = 0, \qquad x, y \in C([-h, 0], \mathbb{R}).$$

For all $\tau \in (0, d]$, we have

$$\left|T_f(x)(\tau) - T_f(y)(\tau)\right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1} \left| f\left(s, x(s), x(g(s))\right) - f(s, y(s), y(g(s)) \right| ds$$

$$\leq \frac{L_f}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\, (\psi(\tau) - \psi(s))^{\alpha-1} (\psi(s) - \psi(0))^{\gamma-1} \left\{ (\psi(s) - \psi(0))^{1-\gamma} [|x(s) - y(s)| + |x(g(s) - y(g(s))|] \right\} ds$$

$$\leq \frac{L_f}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\, (\psi(\tau) - \psi(s))^{\alpha-1} (\psi(s) - \psi(0))^{\gamma-1} \times [\max_{s \in [0,d]} \left| (\psi(s) - \psi(0))^{1-\gamma} x(s) - y(s) \right| + \max_{s \in [0,d]} \left| (\psi(s) - \psi(0))^{1-\gamma} x(g(s) - y(g(s)) \right|]\, ds$$

$$\leq \frac{2L_f}{\Gamma(\alpha)} \|x-y\|_{C_{1-\gamma\,;\,\psi}}[0,d] \int_0^\tau \psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\left(\psi(s)-\psi(0)\right)^{\gamma-1} ds$$

$$= \frac{2L_f\left(\psi(\tau)-\psi(0)\right)^{\alpha+\gamma-1}}{\Gamma(\alpha)} \frac{\Gamma(\gamma)\Gamma(\alpha)}{\Gamma(\gamma+\alpha)} \|x-y\|_{C_{1-\gamma\,;\,\psi}}[0,d],$$

Which implies that

$$\left\|T_f(x)-T_f(y)\right\|_{C_{1-\gamma\,;\,\psi}}[0,d] \leq \frac{2L_f\Gamma(\gamma)\left(\psi(d)-\psi(0)\right)^{\alpha}}{\Gamma(\alpha+\gamma)} \|x-y\|_{C_{1-\gamma\,;\,\psi}}[0,d].$$

Thus T_f is a contraction (via the norm $\|\,.\,\|_{C_{1-\gamma;\psi}}[0,d]$on X). Now apply the Banach contraction principle to establish (i).

Now we prove (ii). Let $y \in C[-h,0] \cap C_{1-\gamma;\psi}[0,d]$ be a solution to (2). We denote by $x \in C[-h,0] \cap C_{1-\gamma;\psi}[0,d]$ the unique solution to problem (1). Now $\varphi(\tau)$, $\tau \in [-h,0]$,

$$x(\tau) = \frac{\left(\psi(\tau)-\psi(0)\right)^{\gamma-1}}{\Gamma(\gamma)} y_0 + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1} f\left(s, y(s), y\left(g(s)\right)\right) ds, \quad \tau \in (0,d].$$

From Remark 2.10 we have

$$\left| y(\tau) - \frac{\left(\psi(\tau)-\psi(0)\right)^{\gamma-1}}{\Gamma(\gamma)} y_0 - \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1} f\left(s, y(s), y\left(g(s)\right)\right) ds \right|$$

$$\leq \varepsilon \mathbb{E}_\alpha\left(\left(\psi(\tau)-\psi(0)\right)^\alpha\right) \qquad (10)$$

for $\tau \in (0,d]$ and note that $|y(\tau)-x(\tau)| = 0$ for $\tau \in [-h,0]$.

For all $\tau \in (0,d]$, it follows from (H_2) and (10) that

$$|y(\tau)-x(\tau)|$$

$$\leq \left| y(\tau) - \frac{\left(\psi(\tau) - \psi(0)\right)^{\gamma-1}}{\Gamma(\gamma)} y_0 - \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1} f\left(s, y(s), y\left(g(s)\right)\right) ds \right| + \left| \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1} f\left(s, y(s), y\left(g(s)\right)\right) ds - \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1} f\left(s, x(s), x\left(g(s)\right)\right) ds \right|$$

$$\leq \varepsilon \mathbb{E}_\alpha\left(\left(\psi(\tau) - \psi(0)\right)^\alpha\right) + \frac{L_f}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1} [|y(s) - x(s)| + |y(g(s) - x(g(s))|] ds . \quad (11)$$

For all $w \in C([-h, d], \mathbb{R}_+)$, consider the operator

$$T_1 : C([-h, d], \mathbb{R}_+) \to C([-h, d], \mathbb{R}_+)$$

defined by

$$T_1(w)(\tau) = \begin{cases} 0, & \tau \in [-h, 0] \\ \varepsilon \mathbb{E}_\alpha\left(\left(\psi(\tau) - \psi(0)\right)^\alpha\right) + \frac{L_f}{\Gamma(\alpha)} \left(\int_0^\tau \psi'(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1} w(s)\, ds + \int_0^\tau \psi'(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1} w\left(g(s)\right) ds \right), & \tau \in (0, d]. \end{cases}$$

We prove that T_1 is a Picard operator. For all $\tau \in [0, d]$, it follows from (H_2) that

$$|T_1(w)(\tau) - T_1(z(\tau))| \leq \frac{2L_f \Gamma(\gamma)\left(\psi(\tau) - \psi(0)\right)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} \|w - z\|_{C_{1-\gamma\,;\,\psi}}[0, d].$$

for all $w, z \in C([-h, d], \mathbb{R})$. Then we obtain

$$\|T_1(w) - T_1(z)\|_{C_{1-\gamma\,;\,\psi}}[0, d] \leq \frac{2L_f \Gamma(\gamma)\left(\psi(d) - \psi(0)\right)^{\alpha}}{\Gamma(\alpha+\gamma)} \|w - z\|_{C_{1-\gamma\,;\,\psi}}[0, d].$$

for all $w, z \in C([-h, d], \mathbb{R})$. Thus T_1 is a contraction (via the norm $\| . \|_{C_{1-\gamma;\psi}}[0, d]$on $C([-h, d], \mathbb{R})$).

Applying the Banach contraction principle to T_1, we see that T_1 is a Picard operator and $F_{T_1} = w^*$. Then , for all $\tau \in [0, d]$, we have

$$w^*(\tau)(= T_1 w^*(\tau))$$

$$= \varepsilon \mathbb{E}_\alpha\left(\left(\psi(\tau) - \psi(0)\right)^\alpha\right) + \frac{L_f}{\Gamma(\alpha)}\left(\int_0^\tau \psi'(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1} w^*(s)\, ds + \int_0^\tau \psi'(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1} w^*\left(g(s)\right) ds\right).$$

Next, we show that the solution w^* is increasing. For all $0 \le \tau_1 \le \tau_2 \le d$ (letting $m := \min_{s\in[0,d]} [w^*(s) + w^*(g(s)] \in \mathbb{R}_+$), we have

$$w^*(\tau_2) - w^*(\tau_1)$$

$$= \varepsilon\left[\mathbb{E}_\alpha\left(\left(\psi(\tau_2) - \psi(0)\right)^\alpha\right) - \mathbb{E}_\alpha\left(\left(\psi(\tau_1) - \psi(0)\right)^\alpha\right)\right] + \frac{L_f}{\Gamma(\alpha)}\int_0^{\tau_1} \psi'(s)\left[\left(\psi(\tau_2) - \psi(s)\right)^{\alpha-1} - \left(\psi(\tau_1) - \psi(s)\right)^{\alpha-1}\right](w^*(s) + w^*(g(s))\, ds + \frac{L_f}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} \psi'(s)\left(\psi(\tau_2) - \psi(s)\right)^{\alpha-1}(w^*(s) + w^*(g(s))\, ds)$$

$$\ge \varepsilon\left[\mathbb{E}_\alpha\left(\left(\psi(\tau_2) - \psi(0)\right)^\alpha\right) - \mathbb{E}_\alpha\left(\left(\psi(\tau_1) - \psi(0)\right)^\alpha\right)\right] + \frac{mL_f}{\Gamma(\alpha)}\int_0^{\tau_1} \psi'(s)\left[\left(\psi(\tau_2) - \psi(s)\right)^{\alpha-1} - \left(\psi(\tau_1) - \psi(s)\right)^{\alpha-1}\right] ds + \frac{mL_f}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} \psi'(s)\left(\psi(\tau_2) - \psi(s)\right)^{\alpha-1} ds$$

$$= \varepsilon\left[\mathbb{E}_\alpha\left(\left(\psi(\tau_2) - \psi(0)\right)^\alpha\right) - \mathbb{E}_\alpha\left(\left(\psi(\tau_1) - \psi(0)\right)^\alpha\right)\right] + \frac{mL_f}{\Gamma(\alpha+1)}\left[\left(\psi(\tau_2) - \psi(s)\right)^\alpha - \left(\psi(\tau_1) - \psi(s)\right)^\alpha\right]$$

$$> 0 .$$

Thus w^* is increasing, so $w^*(g(\tau)) \le w^*(\tau)$ since $g(\tau) \le \tau$ and

$$w^*(\tau) \le \varepsilon \mathbb{E}_\alpha\left(\left(\psi(\tau) - \psi(0)\right)^\alpha\right) + \frac{2L_f}{\Gamma(\alpha)}\int_0^\tau \psi'(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1} w^*(s)\, ds .$$

Using Lemma 2.11 and Remark 2.12, we obtain

$$w^{*}(\tau) \leq \varepsilon \mathbb{E}_{\alpha}\left((\psi(\tau)-\psi(0))^{\alpha}\right) \mathbb{E}_{\alpha}\left(2L_{f}(\psi(\tau)-\psi(0))^{\alpha}\right) \quad (\tau \in [0,d])$$

$$\leq \varepsilon \mathbb{E}_{\alpha}\left((\psi(\tau)-\psi(0))^{\alpha}\right) \mathbb{E}_{\alpha}\left(2L_{f}(\psi(d)-\psi(0))^{\alpha}\right)$$

$$= {}^{c}\mathbb{E}_{\alpha} \varepsilon \mathbb{E}_{\alpha}\left((\psi(\tau)-\psi(0))^{\alpha}\right),$$

Where ${}^{c}\mathbb{E}_{\alpha} \coloneqq \mathbb{E}_{\alpha}\left(2L_{f}(\psi(d)-\psi(0))^{\alpha}\right)$.

In particular, if $w = |y-x|$, from (11), $w \leq T_1 w$ by Lemma 2.8 we obtain $w \leq w^{*}$, where T_1 is an increasing Picard operator. As a result, we get

$$|y(\tau)-x(\tau)| \leq {}^{c}\mathbb{E}_{\alpha} \varepsilon \mathbb{E}_{\alpha}\left((\psi(\tau)-\psi(0))^{\alpha}\right), \qquad \tau \in [-h,d],$$

And thus (2) is Ulam – Hyers – Mittag – Leffler stable.

Now we change (H_3) to

(H_4) We have the inequality

$$\frac{2L_f \Gamma(\alpha) e^{\theta(\psi(d)-\psi(0))}(\psi(d)-\psi(0)^{\alpha}}{\Gamma(\alpha+\gamma)} < 1, \qquad \theta > 0.$$

Theorem 3.2

Assume that (H_1), (H_2), and (H_4) are satisfied. Then

(i) (2) – (4) has a unique solution in $C[-h,d] \cap C_{1-\gamma;\psi}[0,d]$.

(ii) (2) is Ulam – Hyers – Mittag – Leffler stable.

Proof

As in Theorem 3.1, we need only prove that T_f defined as before is a contraction on X (via the norm$\|\,.\,\|_B$). Since the process is standard, we only give the main difference in the proof as follows: For all $\tau \in (0,d]$, we have

$$\left|T_f(x)(\tau)-T_f(y)(\tau)\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{\tau} \psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\left|f\left(s,x(s),x\left(g(s)\right)\right) - f\left(s,y(s),y\left(g(s)\right)\right)\right| ds$$

$$\leq \frac{L_f}{\Gamma(\alpha)} \int_0^{\tau} \psi'(s) \left(\psi(\tau) - \psi(s)\right)^{\alpha-1} (|x(s) - y(s)| + |x(g(s) - y(g(s)|)$$

$$\leq \frac{L_f}{\Gamma(\alpha)} \int_0^{\tau} \psi'(s) \left(\psi(\tau) - \psi(s)\right)^{\alpha-1} \left(\psi(s) - \psi(0)\right)^{\gamma-1} e^{\theta(\psi(s) - \psi(0))} \times \{\max_{0\leq s\leq d} \left(\psi(s) - \psi(0)\right)^{1-\gamma} e^{-\theta(\psi(s) - \psi(0))} (|x(s) - y(s)| + \left|x(g(s) - y\left(g(s)\right)\right|)\}\, ds$$

$$\leq \frac{2L_f}{\Gamma(\alpha)} \|x - y\|_B \int_0^{\tau} \psi'(s) \left(\psi(\tau) - \psi(s)\right)^{\alpha-1} \left(\psi(s) - \psi(0)\right)^{\gamma-1} e^{\theta(\psi(s) - \psi(0))}\, ds$$

$$\leq \frac{2L_f}{\Gamma(\alpha)} \|x - y\|_B e^{\theta(\psi(d) - \psi(0))} \int_0^{\tau} \psi'(s) \left(\psi(\tau) - \psi(s)\right)^{\alpha-1} \left(\psi(s) - \psi(0)\right)^{\gamma-1}\, ds$$

$$\leq \frac{2L_f \Gamma(\alpha) e^{\theta(\psi(d) - \psi(0))} (\psi(d) - \psi(0)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} \|x - y\|_B .$$

Then

$$\left\|T_f(x) - T_f(y)\right\|_B \leq \frac{2L_f \Gamma(\alpha) e^{\theta(\psi(d) - \psi(0))} (\psi(d) - \psi(0)^{\alpha}}{\Gamma(\alpha+\gamma)} \|x - y\|_B .$$

Thus T_f is a contraction (via the norm$\| . \|_B$ $on\ X$) .

4. Examples

In this section, we give two examples illustrating our main results.

Example 4.1

Consider the fractional – order system

$$\begin{cases} {}^{H}\mathbb{D}_{0^+}^{\frac{1}{2},\frac{1}{2};e^{\tau}} x(\tau) = \frac{1}{4}\frac{x^2(\tau-1)}{1+x^2(\tau-1)} + \frac{1}{4} arctan\left(x(\tau)\right), \quad \tau \in (0,1], \\ I_{0^+}^{1-\gamma;e^{\tau}}\, x(0^+) = x_0, \end{cases} \qquad (12)$$

$$x(\tau) = 0, \tau \in [-h, 0],$$

and the following inequality

$$ {}^{H}\mathbb{D}_{0^+}^{\frac{1}{2},\frac{1}{2};e^{\tau}} x(\tau) - f(\tau, y(\tau), y(\tau-1)) \leq \varepsilon E_{\frac{1}{2}}\left((e^{\tau}-1)^{\frac{1}{2}}\right). $$

Let $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$. Then $\gamma = \alpha + \beta(1-\alpha) = \frac{3}{4}$, $d = 1, \psi(\,.\,) = e, g(\,.\,) = .-1$,

$f\left(., x(\,.\,), g(x(\,.\,))\right) = \frac{1}{4}\frac{x^2(\,.-1)}{1+x^2(.-1)} + \frac{1}{4}arctan(x(\,.\,))$, and $L_f = \frac{1}{4}$. Thus,

$$ \frac{2L_f\Gamma(\alpha)(\psi(d) - \psi(0)^{\alpha}}{\Gamma(\alpha+\gamma)} \approx 0.8861 < 1. $$

Now all the assumptions in Theorem 3.1 are satisfied, so problem (12) has a unique solution, and the first equation in (12) is Ulam – Hyers – Mittag – Leffler stable with

$$ |y(\tau) - x(\tau)| \leq c_{\mathbb{E}_{\frac{1}{2}}}\varepsilon\mathbb{E}_{\frac{1}{2}}\left((e^{\tau}-1)^{\frac{1}{2}}\right), \qquad \tau \in [-1, 1], $$

where $c_{\mathbb{E}_{\frac{1}{2}}} = \mathbb{E}_{\frac{1}{2}}\left(\frac{\sqrt{e-1}}{2}\right)$.

Example 4.2

Consider the fractional – order system

$$ \begin{cases} {}^{H}\mathbb{D}_{0^+}^{\frac{1}{3},\frac{1}{4};\tau^2} x(\tau-2) = \frac{1}{5}\frac{x^2(\tau-2)}{1+x^2(\tau-2)} + \frac{1}{5}sin(x(\tau-2)), \quad \tau \in (0, 1], \\ I_{0^+}^{1-\gamma;\tau^2}\, \boldsymbol{x}(0^+) = x_0, \end{cases} \qquad (13) $$

$$ x(\tau) = 0, \tau \in [-1, 0], $$

and the inequality

$$ {}^{H}\mathbb{D}_{0^+}^{\frac{1}{3},\frac{1}{4};\tau^2} x(\tau) - f(\tau, y(\tau), y(\tau-1)) \leq \varepsilon E_{\frac{1}{3}}\left(\tau^{\frac{2}{3}}\right). $$

Following Theorem 3.2, Let $\alpha = \frac{1}{3}$, $\beta = \frac{1}{4}$.Then $\gamma = \alpha + \beta(1-\alpha) = \frac{1}{2}$. Let

$$ d = 1, \quad \theta = \frac{1}{3}, \quad \psi(\,.\,) = .^2, \quad L_f = \frac{1}{5}\text{ . Thus} $$

$$ \frac{2L_f\Gamma(\gamma)e^{\theta(\psi(d)-\psi(0))}(\psi(d) - \psi(0)^{\alpha}}{\Gamma(\alpha+\gamma)} \approx 0.8766 < 1. $$

Now all the assumptions in Theorem 3.2 are satisfied, so problem (13) has a unique solution, and the first equation in (13) is Ulam – Hyers – Mittag – Leffler stable with

$$|y(\tau) - x(\tau)| \le c_{\mathbb{E}_{\frac{1}{3}}} \varepsilon \mathbb{E}_{\frac{1}{3}}\left(\tau^{\frac{2}{3}}\right), \qquad \tau \in [-1, 1],$$

where $c_{\mathbb{E}_{\frac{1}{3}}} = \mathbb{E}_{\frac{1}{3}}\left(\frac{2}{5}\right)$.

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